Lecture 7.

Borel measures on $\mathbb{R}$.

Suppose we start w/ an idea of "measure" of an interval $(a,b) \subseteq \mathbb{R}$.
Typically, we may have $f \in \mathcal{C}(\mathbb{R})$, $f \geq 0$, we want

$$\mu((a,b)) = \int_a^b f(x)\,dx.$$

E.g. $f(x) = 1 \Rightarrow \mu((a,b)) = b-a$.

If we write $F(y)$ for a primitive function $F(y) = \int_0^y f(x)\,dx$, then

$$\mu((a,b)) = F(b) - F(a).$$

(*) = nondecreas. in Folland.

$F(y)$ is then increasing(*), continuous function, "distribution function for μ".

For technical reasons, we shall work w/ the collection $\mathcal{E}$ of half-open intervals $(a,b]$ and (a,∞), where $-\infty \le a < b < \infty$, and we add $\emptyset$. Then $\mathcal{E}$ is an <u>elementary family</u> of sets $E \subseteq X = \mathbb{R}$.

- $\emptyset \in \mathcal{E}$
- $E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$
- $E \in \mathcal{E} \Rightarrow E^c$ is a disjoint, finite union of sets in $\mathcal{E}$.

This is why we use half-open intervals →

By <u>Prop 1.7</u> in Folland, the collection $\mathcal{A}$ of <u>finite, disjoint unions</u> of sets in $\mathcal{E}$ is an <u>algebra</u>.

<u>Prop 1.</u> Let $F: \mathbb{R} \to \mathbb{R}$ be increasing and right continuous ($F(y) = \lim_{x \to y^+} F(y), \forall y$), and for any finite, disjoint union $A = \bigcup_{k=1}^{n} (a_k, b_k]$, set

$$\mu_0(A) = \sum_{k=1}^{n} \left(F(b_k) - F(a_k)\right), \quad \mu_0(\emptyset) = 0$$

Then μ_0 is a premeasure on $\mathcal{A}$.

The decomp. of $\bigcup_{k=1}^{n}(a_k, b_k]$ is not unique.

Pf. ① Is μ_0 well-defined? The order of the intervals is of no importance. Thus, after renumbering if necessary, have $-\infty \le a_1 < a_2 \dots < a_n$. Since they are disjoint

$a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n$. The ambiguity arises if, e.g., $b_1 = a_2$. We can then replace $(a_1, b_1] \cup (a_2, b_2]$ by $(a_1, b_2]$. But then the term

$$\left(F(b_1) - F(a_1)\right) + \left(F(b_2) - F(a_2)\right)$$

$$= F(b_2) - F(a_1).$$

This shows that μ_0 is welldefined on $\mathcal{A}$. Also, μ_0 is finitely additive.

② Suppose now $\{I_k = (a_k, b_k]\}_{k=1}^{\infty}$ are disjoint and $E = \bigcup_{k=1}^{\infty} I_k$ is in $\mathcal{A}$.

Since $\bigcup_{k=1}^{n} I_k \subseteq E \Rightarrow \sum_{k=1}^{n} \mu_0(I_k) = \mu_0(\bigcup_1^n I_k) \leq \mu_0(E)$

$\Rightarrow \sum_{k=1}^{\infty} \mu_0(I_k) \leq \mu_0(E)$.

For opposite ineq., we need right continuity. First, since $E = \bigcup_{j=1}^{n} I_n'$, $I_n' = (a_n', b_n']$, we must have the collection of I_k split into collections whose unions are $(a_k', b_k']$. By finite additivity it suffices to consider $\bigcup_{k=1}^{\infty} (a_k, b_k] = (a, b] =: E$. We shall assume that $a \neq -\infty$ so $\nearrow$ is finite interval. (The case $a = -\infty$ is similar, - see Folland.) A moments reflection shows that for some k, say $k=1$, $b_k = b$. We can then arrange the intervals $I_k = (a_k, b_k]$ s.t. $b_1 = b$, and then $b = b_1 > a_1 = b_2 > a_2 = \dots > a$, and $a_k \searrow a$ as $k \to \infty$. Since F is right cont., for any $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $F(a+\delta) \leq F(a) + \varepsilon$. There is N s.t. $a_n \leq a + \delta$ for $n \geq N$.

$\Rightarrow \mu_0(E) = F(b) - F(a) \leq F(b) - F(a+\delta) + \varepsilon$

$$\le F(b) - F(a_N) + \varepsilon = \sum_{j=1}^{N} \left(F(b_j) - F(a_j) \right) + \varepsilon$$

$$= \sum_{j=1}^{N} \mu_0(I_j) + \varepsilon \quad \Rightarrow \mu_0(E) \le \sum_{j=1}^{\infty} \mu_0(I_j)$$

$$\Rightarrow \mu_0(E) = \sum_{k=1}^{\infty} \mu_0(I_k).$$

An important corollary that follows from Prop 1 and previous results (Thm 1.14):

<u>Cor 1</u>. Let $F: \mathbb{R} \to \mathbb{R}$ be increasing, right cont. fcn. Then $\exists$ unique Borel measure μ_F s.t. $\mu((a,b]) = F(b) - F(a)$ for all $a < b$. If G is incr., right cont. s.t. $\mu_F = \mu_G$ then $F - G$ is constant.

We also have

<u>Thm 1</u>. If μ is Borel measure, then

$$F(x) = \begin{cases} \mu((0,x]), & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]), & x < 0 \end{cases}$$

is increasing and right cont.

Pf. Clearly, F is increasing by monotonicity. Need to check right cont. Assume $b>0$. Then,

$$(0,b]=\bigcap_{k=1}^{\infty}(0,b_n] \text{ , where } b_n \to b^+ .$$

Thus, $F(b)=\lim_{n\to\infty}F(b_n)$, $b_n \to b^+$.
by cont. from above

$\Rightarrow \lim_{x\to b^+} F(x)=F(b)$, i.e. right cont. $\boxtimes$

Rem. We denote by $\bar{\mu}_F$ the completion of the Borel measure μ_F above. $\bar{\mu}_F$ is called Lebesgue-Stieltjes measure of F.

If $F(x)=x$ $\quad (\Rightarrow \mu_F((a,b])=b-a)$

then μ_F is the Lebesgue measure.